Disappeared circle (New Way, Book 2, p.101, No.4)

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Question

Find the equation of the circle passing through the intersections of the circles
$C_1 : x^2 + y^2 = 4$ and $C_2 : x^2 + y^2 + 2x - 4y - 4 = 0$ and having area equal to 9π square units .

Solution 1 (unsatisfactory)

Let the required equation of the circle be:

$C_1 + kC_2$: $\quad x^2 + y^2 - 4 + k(x^2 + y^2 + 2x - 4y - 4) = 0$ .... **(1)**

i.e. $\quad (1 + k)x^2 + (1 + k)y^2 + 2kx - 4ky - (4 + 4k) = 0$

$$x^2 + y^2 + \frac{2k}{1+k}x - \frac{4k}{1+k}y - 4 = 0 \quad \text{.... } \mathbf{(2)}$$

Radius, $r = \sqrt{\left(\frac{k}{1+k}\right)^2 + \left(\frac{2k}{1+k}\right)^2 + 4} = \sqrt{\frac{9k^2 + 8k + 4}{(1+k)^2}}$

Since Area $= \pi r^2 = 9\pi$ $\quad\therefore\quad r = 3$.

$\therefore \quad \sqrt{\frac{9k^2 + 8k + 4}{(1+k)^2}} = 3 \quad \Rightarrow 9k^2 + 8k + 4 = 9(1+k)^2 \quad \Rightarrow 8k + 4 = 9 + 18k \quad \Rightarrow k = -\frac{1}{2}$

$\therefore$ The required equation of the circle is $x^2 + y^2 - 2x + 4y - 4 = 0$ .... **(3)**

Solution 2

Let the required equation of the circle be:

$C_2 + kC_1$: $\quad (x^2 + y^2 + 2x - 4y - 4) + k(x^2 + y^2 - 4) = 0$ .... **(4)**

i.e. $\quad (1 + k)x^2 + (1 + k)y^2 + 2x - 4y - (4 + 4k) = 0$

$$x^2 + y^2 + \frac{2}{1+k}x - \frac{4}{1+k}y - 4 = 0 \quad \text{.... } \mathbf{(5)}$$

Radius, $r = \sqrt{\left(\frac{1}{1+k}\right)^2 + \left(\frac{2}{1+k}\right)^2 + 4} = \sqrt{\frac{4k^2 + 8k + 9}{(1+k)^2}}$

Since Area $= \pi r^2 = 9\pi$ $\quad\therefore\quad r = 3$.

$\therefore \quad \sqrt{\frac{4k^2 + 8k + 9}{(1+k)^2}} = 3 \quad \Rightarrow 4k^2 + 8k + 9 = 9(1+k)^2 \quad \Rightarrow 5k^2 + 10k = 0 \quad \Rightarrow k(k+2) = 0$

$\therefore \quad k = 0$ or -2 . Using **(5)** , we have:

When $k = 0$, the required equation of the circle is $x^2 + y^2 + 2x - 4y - 4 = 0$.

When $k = -2$, the required equation of the circle is $x^2 + y^2 - 2x + 4y - 4 = 0$.

$\therefore$ Two possible eqs: $x^2 + y^2 + 2x - 4y - 4 = 0$ or $x^2 + y^2 - 2x + 4y - 4 = 0$ .... **(6)**

Discussion

As can be seen in **(6)**, one of the solution circles is $C_2 : x^2 + y^2 + 2x - 4y - 4 = 0$.

In **solution 1**, the family of circles $C_1 + kC_2 = 0 \equiv \frac{1}{k}C_1 + C_2 = 0$ does not include C_2, where $k = \infty$. As a result, there is only one circle for **solution 1** as in **(3)**.

(Similarly, $C_2 + kC_1$ does not include C_1)

Solution 3

$C_1 : x^2 + y^2 - 4 = 0$ …. **(7)** $C_2 : x^2 + y^2 + 2x - 4y - 4 = 0$ …. **(8)**

Their common chord is given by $L : x^2 + y^2 - 4 = x^2 + y^2 + 2x - 4y - 4$

or $L : x - 2y = 0$

Let the required equation of the circle be:

$C_1 + kL$: $(x^2 + y^2 - 4) + k(x - 2y) = 0$

i.e. $x^2 + y^2 + kx - 2ky - 4 = 0$ …. **(9)**

Radius, $r = \sqrt{\left(\frac{k}{2}\right)^2 + k^2 + 4} = \sqrt{\frac{5k^2 + 16}{4}}$

Since Area $= \pi r^2 = 9\pi$ $\therefore$ $r = 3$.

$\therefore$ $\sqrt{\frac{5k^2 + 16}{4}} = 3 \Rightarrow 5k^2 + 16 = 36$ $\Rightarrow 5k^2 - 20 = 0$ $\Rightarrow k^2 - 4 = 0$

$\therefore$ $k = 2$ or -2.

From **(9)**,

When $k = 2$, the required equation of the circle is $x^2 + y^2 + 2x - 4y - 4 = 0$.

When $k = -2$, the required equation of the circle is $x^2 + y^2 - 2x + 4y - 4 = 0$.

$\therefore$ Two possible equations: $x^2 + y^2 + 2x - 4y - 4 = 0$ or $x^2 + y^2 - 2x + 4y - 4 = 0$.

Think : How many solution(s) if we use **(a)** $C_2 + kL$, **(b)** $L + kC_1$, **(c)** $L + kC_2$ **?**

Solution 4

You may not use the concept of family of circles to find the solution, but it will be longer. The working scheme is as follows:

1. Solve (7) and (8). The points of intersections are $\left(\pm\frac{4}{\sqrt{5}}, \pm\frac{2}{\sqrt{5}}\right)$.
2. Let $G(a, b)$ be the centre of the required circle.
   The distance between G and one of the two points of contact $= r = 3$.
   You set up two equations involving (a, b).
3. Solve, you can get the centres of possible circles : $(1, -2)$ or $(-1, 2)$
4. Using centre-radius form, the required circles are:
   $x^2 + y^2 + 2x - 4y - 4 = 0$ or $x^2 + y^2 - 2x + 4y - 4 = 0$.